

# INTEGRAL GEOMETRY AND MIZEL'S PROBLEM

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*Dedicated to professor Lech Górniewicz in occasion of his 70 birthday*

## 1. Introduction

A subject treating in this article combines in one bundle some questions of complex analysis, geometry and probability theory. First investigations of geometric probability start from well known Buffon's needle problem and related paradoxes. Let a needle be considered as a real line, and then the problem reduces to finding some invariant measure of set relative to movement (L. Santalo, G. Matheron, R. V. Ambarcumjan [1, 8, 12]).

Many questions of integral geometry are reduced to estimation of the measure linear spaces crossing convex set. The finding of such measure allows to the probabilistic estimations. Other problems more close to geometry are estimations properties of set under investigation if properties of its intersections with families of some sets are well known:

- 1) with planes of fixed dimension:
  - a) real case (G. Aumann, A. Kosiński, E. Shchepin [2, 7, 13]);
  - b) complex case (Yu. Zelinskii [18]);
- 2) with a set of vertices of an arbitrary rectangle (A. Besicovitch, L. Danzer, T. Zamfirescu, M. Tkachuk [3, 4, 17, 14]).

First problem is connected with the well known Ulam's problem from Scottish book [9].

**Ulam problem.** Let  $M^n$  be  $n$ -dimensional manifold and every section of  $M^n$  by arbitrary hyperplane  $L$  be homeomorphic to  $(n - 1)$ -dimensional sphere  $S^{n-1}$ . Is it true that  $M^n$  is  $n$ -dimensional sphere?

In the real case A.Kosiński solved this problem in 1962 [7]. L.Montejano received the repetition of this result in 1990 [10]. In complex case Yu.Zelinskii received similar result in 1993 [18].

The second problem is known in literature as Mizel's problem. Below review of results related to this problem will be given and new unsolved problems in this direction are discussed.

## 2. Mizel problem

**Mizel problem** (Characterization of a circle). A closed convex curve such that, if three vertices of any rectangle lie on it, so does the fourth, must be a circle.

In 1961, Besicovitch [3] solved this problem. Later modified proof of this statement was presented by Danzer [4], Watson [16], Koenen [6], Nash-Williams [11].

In 1989, T. Zamfirescu [17] proved the similar result for Jordan curve (not convex a priori) and for rectangle with infinitesimal relation between sides.

$$\left| \frac{ac}{ab} \right| \leq \varepsilon > 0.$$

In 2006, M. Tkachuk [14] received the most general result in this area for an arbitrary compact set  $C \subset \mathbb{R}^2$ , where the complement  $\mathbb{R}^2 \setminus C$  is not connected.

Obviously that requirement to compactness is necessary, otherwise straight line and some other sets, noted below, will satisfy the assessed requirement. But if not to require partition to planes, the following ensembles will satisfy, for instance, that condition of the rectangle : ensemble from three points of plane such that triangle with vertices in these points will not be rectangular, a proper arc of semicircle, set points of plane with rational (irrational) coordinates.

Zamfirescu in his article [17] invited his readers to verify whether indeed an arbitrary convex curve  $\Gamma$  of constant width  $d$  satisfying the infinitesimal rectangular property is a circle. In this paper we shall give an affirmative answer to this question in the following theorem:

**Theorem 1.** The convex curve of constant width satisfying the infinitesimal rectangular condition is a circle.

Combining this result with the result of Zamfirescu [17] we obtain:

**Theorem 2.** Every Jordan curve satisfying infinitesimal rectangle condition is a circle.

We are only interested rectangles with the diagonal length  $d$ , so the infinitesimal condition may be replaced by the requirement that the smaller rectangle side has length less than some  $\varepsilon$ .

In [14] it was proved that the curve  $\Gamma$  has a continuous tangent.

We introduce the following notation: at each point  $x \in \Gamma$  denote by  $\partial\Delta_x$  the circle having with the curve  $\Gamma$  a common tangent at the point  $x$ ; in some neighborhood of a point of the curve  $\Gamma$  we assume that upward direction is the direction along the the inner normal and according with this we consider the right direction and the left direction along the curve  $\Gamma$ ;  $U_\varepsilon(x, \Gamma)$  is the  $\varepsilon$ -neighborhood of  $x$  on the curve  $\Gamma$ ,  $U_\varepsilon^l(x, \Gamma)$  is the left  $\varepsilon$ -neighborhood of  $x$  on the curve  $\Gamma$ , i.e. the subset of  $U_\varepsilon(x, \Gamma)$  each point of which lies to the left of  $x$ ;  $U_\varepsilon^r(x, \Gamma)$  is the right  $\varepsilon$ -neighborhood of  $x$  on  $\Gamma$ .

Having in the proof provided by Besicovitch [3] a local character all four lemmas are valid for the infinitesimal rectangle property. But the result following from these lemmas changes: if at some point  $x \in \Gamma$  the circle  $\partial\Delta_x$  intersects  $\Gamma$  at  $y$  and the distance between  $x$  and  $y$  is less than  $\varepsilon$ , then in some neighborhood of  $x$  (and also of the opposite point  $x^*$ ) the curve  $\Gamma$  is an arc of the circle  $\partial\Delta_x$ .

Consequently all points of the curve  $\Gamma$  can be divided into five disjoint sets:

$A = \{x \in \Gamma | U_\varepsilon(x, \Gamma) \setminus \{x\} \subset \Delta_x\}$  is the set of points of the curve  $\Gamma$ , such that the curve  $\Gamma$  lies in the open circular disk  $\Delta_x$  in  $\varepsilon$ -neighborhood of  $x \in A$  except the point  $x$  itself;

$B = \{x \in \Gamma | U_\varepsilon(x, \Gamma) \setminus \{x\} \subset \mathbb{R}^2 \setminus \overline{\Delta_x}\}$  is the set of points of the curve  $\Gamma$  such that in their  $\varepsilon$ -neighborhood the curve  $\Gamma$  is situated outside of the closed circular disk  $\overline{\Delta_x}$  except the point  $x$  itself;

$AB = \{x \in \Gamma | U_\varepsilon^l(x, \Gamma) \subset \Delta_x, U_\varepsilon^r(x, \Gamma) \subset \mathbb{R}^2 \setminus \overline{\Delta_x}\}$  is the set of points of the curve  $\Gamma$  such that in their  $\varepsilon$ -neighborhood the curve  $\Gamma$  lies in the open circular disk  $\Delta_x$  to the left of  $x$  and outside of the closed circular disk  $\overline{\Delta_x}$  to the right of  $x$  except the point  $x$  itself;

$BA = \{x \in \Gamma | U_\varepsilon^l(x, \Gamma) \subset \mathbb{R}^2 \setminus \overline{\Delta_x}, U_\varepsilon^r(x, \Gamma) \subset \Delta_x\}$  is the set of points of the curve  $\Gamma$  such that in their  $\varepsilon$ -neighborhood the curve  $\Gamma$  lies in the open circular disk  $\Delta_x$  to the right of  $x$  and outside of the closed circular disk  $\overline{\Delta_x}$  to the left of  $x$  except the point  $x$  itself;

$C$  is the set of points of the curve  $\Gamma$  each of which has the neighborhood where curve coincides with an arc of the circle  $\partial\Delta_x$ .

Hence  $\Gamma = A \cup B \cup AB \cup BA \cup C$ .

Suppose that  $x_n \rightarrow x$ ,  $x_n \in \Gamma$  and let all  $x_n$  be in  $\varepsilon$ -neighborhood of  $x$  to the left of it. Assume that  $x_n \in A \cup BA$  and  $x \in AB$  then in some neighborhood of  $x$  the curve  $\Gamma$  lies above each circle  $\partial\Delta_{x_n}$ .  $\Gamma$  is inside of the circular disk  $\overline{\Delta_x}$ . This fact follows from the continuity of the tangent to  $\Gamma$ . We obtain a contradiction with  $x \in AB$ . So in some neighborhood of  $x$  to the left of it there are no points of the sets  $A$  and  $BA$ . In this neighborhood we consider the similar sequence of points  $x_n \in B \cup AB$  and as a consequence we find that in some neighborhood of  $x$  the curve  $\Gamma$  lies under every circle  $\partial\Delta_{x_n}$  and therefore outside of the circular disk  $\Delta_x$  which is impossible. So in some left half-neighborhood of  $x$  our curve is an arc of a circle  $\partial\Delta_x$  which is also not compatible with the condition  $x \in AB$ . We conclude that  $AB = \emptyset$ . Similarly  $BA = \emptyset$ .

Thus  $\Gamma = A \cup B \cup C$ . Suppose that  $C = \emptyset$ ,  $\Gamma = A \cup B$ . Consider the sequence of points

$x_n \in A$ ,  $x_n \rightarrow x$ . If  $x \in B$  then as we know the contradiction is obtained. Hence  $x$  does belong to  $A$  and  $A$  is a closed set. Similarly  $B$  is closed. The curve  $\Gamma$  is connected and therefore either  $A$  or  $B$  is empty. It means that  $\Gamma = A$  or  $\Gamma = B$  which implies that the curve  $\Gamma$  either contains a circular disk  $\Delta_x$  or is contained in a circular disk  $\Delta_x$  and it contradicts the fact that  $\Gamma$  is the convex curve of constant width and has a length  $\pi d$ .

Hence  $C \neq \emptyset$  and there exists a point  $x \in C$ . Some neighborhood of  $x$  is also contained in the set  $C$ . We shall move along the curve  $\Gamma$  to the left from the point  $x$  to the first point  $y$  that does not belong to  $C$ . It is obviously that  $y$  cannot belong neither to  $A$  nor to  $B$  and therefore  $\Gamma = C$ . Then by applying the Heine-Borel lemma we conclude that  $\Gamma$  is a circle and theorem 1 is proved.

The row of similar opened problems in the plane and in  $n$ -dimensional case appears in connection with Mizel's problem.

**Problem 1.** Let  $C$  be closed Jordan curve in  $\mathbb{R}^2$  and for arbitrary algebraic closed curve  $L$  of order  $n$  from property that intersection  $C \cap L$  contains  $m$  points follows, that  $C \cap L$  contains no less then  $m + 1$  points. Does there exists a number  $m$ , that from property above follows that  $C$  be algebraic curve of order  $n$ ?

**Problem 2.** Let in previous question  $L$  be a circle and  $m = 3$ , is it true, that  $C$  also be a circle?

**Problem 3.** Let in problem 1  $L$  be an ellipse and  $m = 4$ , is it true, that  $C$  also be an ellipse?

**Problem 4.** Will be a compact  $C$  a sphere in  $\mathbb{R}^n$ , if  $C$  divides the space, and if from the belonging  $n + 1$  tops of the arbitrary rectangular parallelepiped to a compact  $C$ , it follows that one more top lies in  $C$  too?

Last question is interesting even if  $C$  be  $(n - 1)$ -dimensional manifold or boundary of a convex set.

**Problem 5.** Let  $C$  be a  $(n - 1)$ -dimensional manifold (or boundary of a convex domain) in  $\mathbb{R}^n$  and not exist  $(n - 1)$ -dimensional sphere  $S^{n-1}$  that intersection  $C \cap S^{n-1}$  contains  $n + 1$  points exactly. Is it true, that  $C$  be a  $(n - 1)$ -dimensional sphere?

**Corollary.** On two-dimensional plane  $\mathbb{R}^2$  there exist compact sets, which divide the plane and such that no circles has with them in accuracy three crosses point. In particular class of similar sets includes the Shottka sets and the Sierpiński's carpet.

**Problem 6.** Remain or not result of [14] true, if we will consider compact set  $C \subset \mathbb{R}^2$ , where the complement  $\mathbb{R}^2 \setminus C$  is not connected?

**Problem 7.** Are cited results and problems 1-6 true, if we will consider that one point (top) on  $C$  is fixed?

Next examples will show that in problems 1-3 it is impossible instead of curve, in analogy with Tkachuk's result, consider compact set dividing plane.

**Example.** We shall consider the domain  $D$  on plane, bounded by circle  $S^1$ . The Surge from it thick in the domain  $D$  infinite ensemble of opened balls  $D_i$ , which are not intersect pair wise even on border and also not intersect the circle  $S^1$ .

Then we receive fractal compact set  $K = \bar{D} \setminus \cup D_i$  without interior points, which divides plane on countable set of components. But it is easy to see that arbitrary circle can intersect  $K$  on only one point or on infinite number of points (see Figure 1).

Other examples we will receive if domains  $D(D_i)$  be domains bounded by ellipses or squares. In this case intersections with circle or ellipse can contain one, two, four or infinite many points but never three points (see Figures 2 and 3). Those examples give negative answer to problem 8 from [19].

Other close problems possible to find in the work of Grünbaum [5].

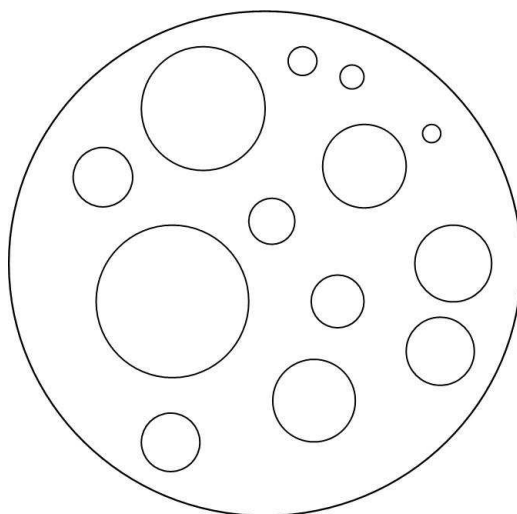


Figure 1

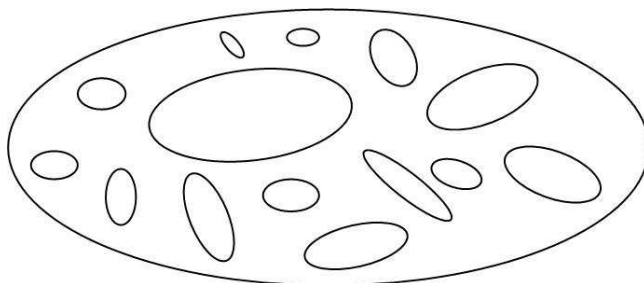


Figure 2

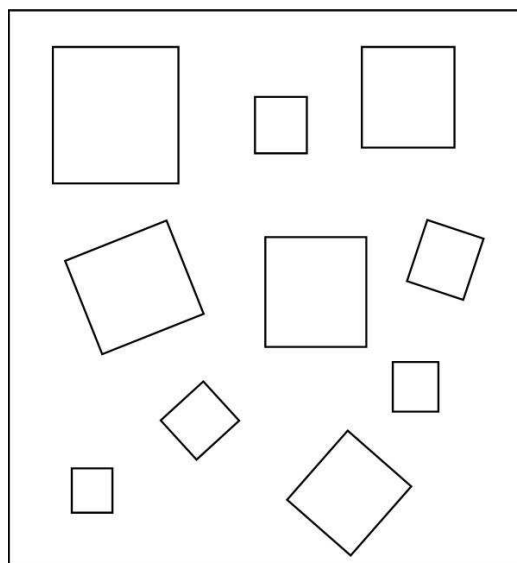


Figure 3

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